Quasi-Static Discrete Green's Function for SAW Devices

David P. Morgan, *Member, IEEE*

***Abstract*—A discrete Green's function is derived for surface acoustic wave device analysis, giving the currents in an infinite array of regular electrodes when a voltage is applied to one electrode. Simple formulas are obtained by using the quasi-static approximation, which assumes the frequency to be such that electrode reflections are insignificant. The function exhibits distortion associated with the nonuniform charge distribution, and is shown to be causal.**

***Index Terms*—Charge distribution, quasi-static approximation, SAW device analysis.**

I. Introduction

THE discrete Green's function (DGF) is a concept developed for analysis of a variety of surface acoustic wave (SAW) and leaky-SAW devices. It is defined by considering an infinite array of identical electrodes, with width a and pitch p, on the surface of a piezoelectric half-space, as shown in Fig. 1. Here, the surface normal is taken as the z-direction, and the SAW propagation direction is x. One electrode, taken to be centred at $x=0$, has unit voltage applied (multiplied by an implicit term $\exp(j\omega t)$, which is omitted), while all other electrodes have zero voltage. The current taken by electrode n is $I_n(\omega)$, and this is equal to the DGF, denoted $G_n(\omega)$. Here, $n=0$ for the electrode centred at $x=0$. Thus, the electrode currents and voltages are

$$I_n = G_n(\omega) \quad V_0 = 1 \quad V_n = 0, \qquad \text{for } n \neq 0. \tag{1}$$

The electrode currents for any other distribution of electrode voltages are obtained simply by superposition. Thus, devices such as transducers and resonators are easily analyzed once the DGF is known, provided the electrodes are regular.

A related function is the harmonic admittance $A_h(\gamma,\omega)$. This is defined by assuming electrode voltages of the form $V_n = V_0\exp(-j2\pi\gamma n)$, where γ is a constant. Then, $A_h(\gamma,\omega) = I_n/V_n$, where I_n are the electrode currents; this ratio can be shown to be independent of n. It is easily shown that $A_h(\gamma,\omega)$ is periodic in γ with period unity, and it can, therefore, be represented by a Fourier series whose coefficients are found to be $G_n(\omega)$. Conversely, $G_n(\omega)$ is a Fourier integral of $A_h((\gamma,\omega)$.

These basic relationships were first developed by Zhang *et al.* [1] and further exploited by, for example, Hashimoto and Yamaguchi [2], [3] and Ventura *et al.* [4]. The main motivation for the approach concerns situations in which complex physical phenomena, difficult to analyze even by numerical methods, need to be accounted for. These include the presence of leaky waves or bulk waves, and the mechanical behavior of finite-thickness electrodes—complications that affect, for example, leaky-wave resonators for RF filters. For such cases, the harmonic admittance affords some degree of simplification because of the assumed form of the electrode voltages. The general procedure is to calculate the harmonic admittance first, by numerical techniques, and then integrate to transform it to the DGF. Care is needed to deal with a pole of $A_h(\gamma,\omega)$, which is often present because of unattenuated surface-wave propagation. Zhang *et al.* [1] introduced a component of the DGF, proportional to $\exp(-js|n|)$ for $n \neq 0$, to algebraically account for the pole in $A_h(\gamma,\omega)$ at $\gamma = s$; here, s is a constant independent of x, though it does depend on ω. Hashimoto and Yamaguchi [2], [3] derived an acoustic DGF of similar form, to which electrostatic terms are added.

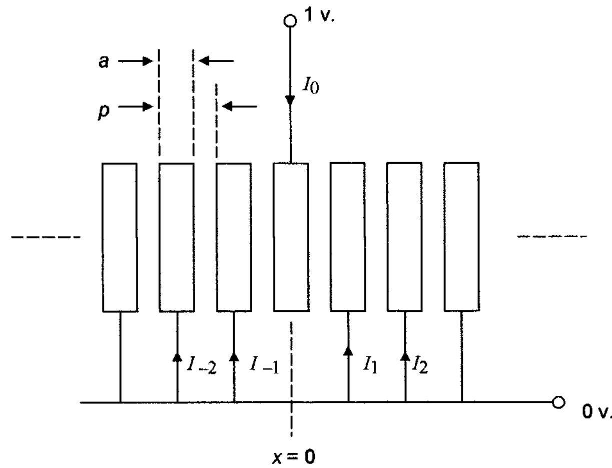

Fig. 1. Electrode configuration for the DGF.

Another application of the DGF has been to the analysis of floating electrode unidirectional transducers (FEUDTs) [2], with regular electrodes. Here, directivity is obtained by means of an asymmetric electrode configuration including one or more floating electrodes in each period. The wave concerned is usually the familiar piezoelectric Rayleigh wave, and the complications mentioned above are of little concern. Moreover, electrode reflections are not usually very significant since they occur at frequencies well above the FEUDT passband. For example, a six-electrode FEUDT (six electrodes per period) has its fundamental passband centred at $f_s/6$, where the sampling frequency f_s is the frequency at which the wavelength equals the electrode spacing p. Electrode reflections occur at the Bragg frequency $f_s/2$ and its multiples. In these circumstances, the mechanical and electrical loading produced by the electrodes

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The author is with Impulse Consulting, Northampton NN3 3BG, U.K.
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do not need to be considered, except in so far as they cause some velocity perturbation. This implies that a simple Green's function analysis using the quasi-static approximation is adequate [5], though the development in [5] does not allow for the presence of floating electrodes. To allow for this, a quasi-static FEUDT analysis was derived in [6], where it was used to evaluate the center-frequency coupled-mode (COM) parameters.

In this paper, a simplified DGF is derived from the quasi-static approximation. The results here show some distortion of the DGF for electrodes with small $|n|$. This is expected because the static charge density, which acts as a distributed SAW source, is spread out over several electrodes on either side of electrode 0 so that the SAW amplitude will vary in this region.

The DGF can be applied to FEUDT analysis by writing equations for the electrode currents in terms of voltages, as explained by Hashimoto and Yamaguchi [2]. Using (1) with superposition, and taking the electrode voltages to be V_m, the electrode currents are

$$I_n = \sum_{m=1}^{M} V_m G_{n-m}. \tag{2}$$

A finite sum is written here, assuming that electrodes with index outside the range $(1,\ M)$ have zero voltage. If the voltages are known, the currents are given directly by (2), and summation of the currents for electrodes connected to one bus bar gives the transducer current and, hence, its admittance. For the FEUDT, the unknown voltages of the floating electrodes can be obtained from (2) and the condition that their currents are zero, leading to simultaneous equations, which determine the voltages. Moreover, the amplitudes of the SAWs generated are, in the quasi-static approximation, easily deduced from the electrode voltages, thus giving the transduction strength and directivity of the transducer. Equation (2) can also be applied to a two-transducer device to obtain the admittance matrix Y_{ij} [2]. With a voltage applied to transducer 1, say, with transducer 2 shorted, the current taken by transducer 1 gives Y_{11} and the current taken by transducer 2 gives Y_{21}.

When applied to FEUDT analysis, the DGF can allow for the frequency variation of the transduction and reflection mechanisms, and can also allow for finite-length effects, unlike the COM analysis in [6]. The assumption of an infinite array of electrodes is not, in practice, very restrictive because electrodes remote from the active part of a transducer have little effect in the quasi-static case. A small number of "guard" electrodes at each end is sufficient to represent the infinite array to adequate accuracy.

II. Quasi-Static Approximation

The Green's function method proposed in [5] and [7] starts from the relation

$$\phi(x,\omega) = G(x,\omega) * \sigma(x,\omega) \tag{3}$$

where $\phi(x,\omega)$ is the surface potential, $\sigma(x,\omega)$ is the charge density on the electrodes, and $G(x,\omega)$ is the Green's function. The asterisk indicates convolution with respect to x. Mechanical perturbations due to the electrodes are ignored. Assuming that the main acoustic activity is a piezoelectric Rayleigh wave, Milson *et al.* [8] showed that we can write $G(x,\omega) = G_e(x) + G_s(x,\omega)$, where $G_e(x)$ is an electrostatic term and $G_s(x,\omega)$ is a term due to the SAWs. A bulk-wave term is omitted on the assumption that its effects can be ignored. The two Green's functions are $G_e(x) = -\ln\ |x|/(\pi\varepsilon)$ and $G_s(x,\omega) = j\Gamma_s \exp(-jk|x|)$, where ε is the surface permittivity, $\varepsilon \equiv (\varepsilon_0 + \varepsilon_p^T) \approx \varepsilon_s(\infty)$, and $k = \omega/v_0$ is the free-surface SAW wavenumber, v_0 being the free-surface velocity. Γ_s is defined as $\Gamma_s = (\Delta v/v)/\varepsilon$, where $\Delta v/v = (v_0 - v_m)/v_0$ and v_m is the SAW velocity for a metallized surface.

The charge density is written as $\sigma(x,\omega) = \sigma_e(x) + \sigma_a(x,\omega)$, where the real function $\sigma_e(x)$ is an electrostatic term obtained by solving (3) with the SAW Green's function $G_s(x,\omega)$ ignored, i.e., taking $G(x,\omega) = G_e(x)$. The second term, i.e., $\sigma_a(x,\omega)$, arises from the acoustic waves present. Substituting for $G(x,\omega)$ and $\sigma(x,\omega)$ in (3) gives four terms on the right-hand side, but a term $G_s(x,\omega) * \sigma_a(x,\omega)$ is relatively small because both quantities are proportional to $\Delta v/v$. In the quasi-static approximation, this term is omitted so that the surface potential becomes

$$\phi(x,\omega) = \left[G_e(x) + G_s(x,\omega)\right] * \sigma_e(x) + G_e(x) * \sigma_a(x,\omega). \tag{4}$$

The surface waves generated by a transducer arise from the $G_s * \sigma_e$ term because the other terms involve the electrostatic Green's function and, therefore, give localized potentials. For a location x outside the transducer, this potential, which can be regarded as the SAW amplitude, is given by

$$\begin{aligned}\phi_s(x,\omega) &= G_s(x,\omega) * \sigma_e(x)\\ &= j\Gamma_s \int_{-\infty}^{\infty} \exp\left[\pm jk(x-x')\right]\sigma_e(x')\,dx'\\ &= j\Gamma_s \overline{\sigma}_e(\pm k)\exp(\pm jkx)\end{aligned} \tag{5}$$

where $\overline{\sigma}_e(k)$ is the Fourier transform of $\sigma_e(x)$. The upper (lower) sign applies when x is to the left-hand side (right-hand side) of the transducer. This shows that for each location x' in the transducer, the electrostatic charge density can be regarded as a source generating SAWs that travel out of the transducer with velocity v_0, unaffected by the electrodes that they pass under. Thus, the stop bands and the velocity perturbation due to the electrodes are not predicted, but these limitations are often acceptable, as explained earlier.

For the case where unit voltage is applied to one electrode in an infinite periodic array, as in Fig. 1, the electrostatic charge density $\sigma_e(x)$ is the elemental charge density denoted $\rho_f(x)$, and the Fourier transform of this has the algebraic form

$$\overline{\rho}_f(k) = \frac{2\varepsilon \sin \pi s}{P_{-s}(-\cos\Delta)} P_m(\cos\Delta), \quad \text{for } m \le kp/(2\pi) \le m+1 \tag{6}$$

where $\Delta = \pi a/p$, $s = kp/(2\pi) - m$ (thus, $0 < s < 1$), $P_m(x)$ is a Legendre polynomial, and $P_{-s}(x)$ is a Legendre function. This function serves as an element factor, governing the SAW generation when a voltage is applied to one electrode; it was first derived by Peach [9].

We can use this function to deduce the DGF for large n. For the situation in Fig. 1, the SAW generated is given by (5) with $\overline{\sigma}_e(k)$ replaced by $\overline{\rho}_f(k)$ so the wave generated in the $+x$-direction is $\phi_s(x,\omega) = j\Gamma_s\overline{\rho}_f(-k)\exp(-jkx)$. This formula gives the wave amplitude outside the "transducer," but here, the "transducer" is actually of infinite length. However, the formula is valid for large finite x because $\rho_f(x)$ is a well-localized function—after a few electrodes the values of $\rho_f(x)$ become small and have little effect on the SAW amplitude. The current entering electrode n is found from the analysis of a receiving transducer. Taking all the electrodes to be shorted and assuming a SAW with potential $\phi_{i1}\exp(-jkx)$ to be incident, the current in electrode n is $I_n = -j\omega\phi_{i1}\overline{\rho}_f(k)$, taken from [5, eq. (4.68)]. Here, the phase of ϕ_{i1} has been referenced to the center of electrode n, which is located at $x_n = np$. Unit aperture is assumed. Identifying ϕ_{i1} with $\phi_s(x_n,\omega)$, the current is found to be

$$I_n \equiv G_n(\omega) = \omega\Gamma_s\left[\overline{\rho}_f(k)\right]^2\exp(-jkx_n), \qquad \text{for } n \gg 1 \tag{7}$$

where we have also used the symmetry $\overline{\rho}_f(-k) = \overline{\rho}_f(k)$.

III. DGF

To derive the DGF for all n, a more detailed analysis is needed. Firstly, we note that the electrode current must have electrostatic and acoustic contributions, corresponding to the two charge densities in (4). The electrostatic charge density is $\rho_f(x)$, and the integral of this over electrode n is the net electrode charge, denoted Q_n. The resulting current is $j\omega Q_n$. Assuming unit aperture, the total electrode current is

$$I_n = I_{an} + j\omega Q_n \tag{8}$$

where I_{an} is the acoustic contribution. The charges Q_n are given by [5, eq. (C.27)] and, in the particular case of $a/p = 1/2$, they are given by the simple formula

$$Q_n = \frac{4\varepsilon}{\pi(1-4m^2)}. \tag{9}$$

The acoustic current is obtained from the relation

$$I_{an} = -j\omega\int_{-\infty}^{\infty}\rho_f(x-x_n)\phi_a(x,\omega)\,dx \tag{10}$$

which is taken from [5, eq. (4.43)]. Here, $\phi_a(x,\omega)$ is the "acoustic potential," which is defined by

$$\phi_a(x,\omega) = G_s(x,\omega) * \rho_f(x) \tag{11}$$

i.e., the acoustic part of the potential in (4), with $\sigma_e(x)$ replaced by its present version $\rho_f(x)$. In view of the form of $G_s(x,\omega)$, $\phi_a(x,\omega)$ is essentially the Fourier transform of the electrostatic charges on one side of the point x, plus the inverse transform of the charges on the other side; this corresponds to the amplitudes of the two waves traveling toward the observation point. An alternative form for ϕ_a can be derived by using the step function, giving the result of [5, eq. (4.47)]. For the present case

$$\phi_a(x,\omega) = \frac{1}{2}j\Gamma_s e^{-jkx}\left[\overline{\rho}_f(-k) + jF(-k)/\pi\right] + \frac{1}{2}j\Gamma_s e^{jkx}\left[\overline{\rho}_f(k) - jF(k)/\pi\right] \tag{12}$$

where the function $F(k)$ is defined by the convolution

$$F(\beta) = \overline{\rho}_f(\beta) * \left[\exp(-j\beta x)/\beta\right]. \tag{13}$$

Making use of $\overline{\rho}_f(-k) = \overline{\rho}_f(k)$, the current I_{an} obtained by substituting (12) into (10) is found to have a real part $g_n(k)$ derived from the $\overline{\rho}_f(\pm k)$ terms, and an imaginary part $jb_n(k)$ derived from the $F(\pm k)$ terms. Thus, we find

$$I_{an} = g_n(k) + jb_n(k) \tag{14}$$

with

$$g_n(k) = \omega\Gamma_s\left[\overline{\rho}_f(k)\right]^2\cos(kx_n) \tag{15}$$

and

$$b_n(k) = -\frac{\omega\Gamma_s}{\pi}\int_{-\infty}^{\infty}\frac{\left[\overline{\rho}_f(k-\gamma)\right]^2}{\gamma}\cos\left[(k-\gamma)x_n\right]\,d\gamma \tag{16}$$

and the DGF is $G_n(\omega) = I_{an}(k) + j\omega Q_n$.

Fig. 2 shows I_{an} for a frequency $f = f_s/6 = v_0/(6p)$, appropriate for the center frequency of a six-electrode FEUDT. The amplitude and phase are defined as $|g_n(k) + jb_n(k)|$ and $\tan^{-1}[b_n(k)/g_n(k)]$, respectively. For clarity, a term $-knp$ has been subtracted from the phase since (6) shows that this is the phase expected for large n. The amplitude has been normalized to $\omega\Gamma_s$. The acoustic Green's function is seen to have constant amplitude and linear phase, except for a few electrodes in the vicinity of electrode 0 where there is notable distortion. The Green's function is, of course, symmetric: thus, $G_{-n}(\omega) = G_n(\omega)$.

Noting that $k = \omega/v_0$, (15) and (16) show that $b_n(k)$ is essentially the Hilbert transform of $g_n(k)$, this transform being the convolution with $-1/(\pi\omega)$ so that the transform of a function $A(\omega)$ is

$$\text{H.T.}\left[A(\omega)\right] = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{A(\omega-\omega')}{\omega'}\,d\omega'. \tag{17}$$

It is well known that a function whose imaginary and real parts are related by this transform will be causal. Here, it can be seen that the function $b_n(k)/\omega$ is the Hilbert transform of $g_n(k)/\omega$. However, multiplication by ω is of little consequence because the phase $\theta = \tan^{-1}[b_n(k)/g_n(k)]$ is not affected, and this phase determines the group delay of the current at frequency ω. Hence, the acoustic current $g_n + jb_n$ is a causal function, as expected because the current in electrode n cannot anticipate the voltage applied to electrode 0.

For large n, the cosine in (16) varies rapidly and the other ω- and k-dependent terms can be treated as constants in comparison. The integral then becomes the Hilbert transform of

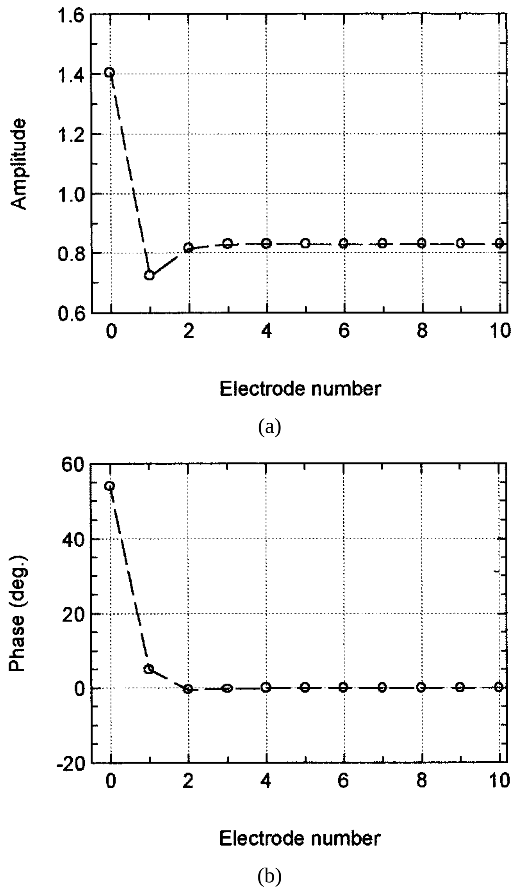

Fig. 2. Acoustic part of the DGF for frequency $f = f_s/6$. (a) Amplitude divided by $\omega\Gamma_s$. (b) Phase relative to $-kx_n$.

$\cos(kx_n)$, which is simply $-\sin(kx_n)$ when $x_n > 0$. This gives $b_n(k) = -\omega\Gamma_s[\overline{\rho}_f(k)]^2 \sin(kx_n)$ and, hence,

$$I_{an} \approx \omega\Gamma_s \left[\overline{\rho}_f(k)\right]^2 \exp(-jkx_n), \qquad \text{for } n \gg 1. \tag{18}$$

Bearing in mind that the electrostatic term is small for large n, this is in agreement with (6).

As a further confirmation, the power flowing into electrode 0 is $g_0/2$, and this should be accounted for by the SAW powers generated. The power of each of the two SAWs is $1/4$ $\omega|\phi_s|^2/\Gamma_s$, and for large $|x|$, the amplitude is $|\phi_s| = \Gamma_s|\overline{\rho}_f(k)|$. This gives $g_0 = \omega\Gamma_s[\overline{\rho}_f(k)]^2$, which is in agreement with (15). It is perhaps worth noting that $|I_n| = \mathrm{Re}[I_0] = g_0(k)$ for $n \gg 1$.

IV. Conclusions

A DGF for SAW propagation has been derived, giving the electrode currents when one electrode in an infinite array has a voltage applied with all the other electrodes grounded. The derivation makes use of the quasi-static approximation and, therefore, ignores reflections and velocity perturbations caused by the electrodes, but these limitations are acceptable for frequencies remote from the stopbands, as in the case of FEUDT analysis. The use of the quasi-static approximation gives sufficient simplification for the DGF to be deduced algebraically. The result shows distortion associated with the form of the charge distribution due to the voltage on one electrode, and several expected features are confirmed, including causality.

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David P. Morgan (S'67–M'72) received the B.A. degree in Physics from Cambridge University, Cambridge, U.K., in 1962, and the M.Sc. degree in solid-state plasmas and the Ph.D. degree in electrical engineering from London University, London, U.K., in 1966 and 1969, respectively. His doctoral research concerned SAW pulse compression.

Since 1969, he has been involved in research and development in a wide variety of topics, mostly in SAWs with the Nippon Electric Company, Kawasaki, Japan (1970–1971), University of Edinburgh, Edinburgh, Scotland (1971–1977), and Plessey Research Caswell, Northampton, U.K. (1977–1986), where he was Group Leader for SAWs. He is currently a consultant in this area. He has authored the textbook *Surface Wave Devices for Signal Processing* (Amsterdam, The Netherlands: Elsevier, 1985, reprinted 1991, Russian-language edition 1990), which summarizes his work on many devices and on the theory of SAW transducers and multistrip couplers. He has authored or co-authored over 80 technical papers on topics including solid-state plasmas, integrated optics, and many SAW areas, including quasi-static and other techniques for transducer analysis, matched filtering for phase-shift keying (PSK) and chirps [interdigital and reflective array compressors (RACs)], bandpass filters, convolvers, distributed acoustic reflection transducers (DARTs), FEUDTs, transverse-coupled filters, and resonators. His knowledge of the SAW area has led to his being invited to lecture on this subject in the U.S., Russia, Finland, Japan, China, and Korea.